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THE SHAPE OF SLENDER THREE-DIMENSIONAL BODIES WITH MAXIMUM DEPTH OF PENETRATION INTO DENSE MEDIA†

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A theory is constructed for optimum slender three-dimensional bodies having maximum depth of penetration under given conditions of entry into a dense medium along the normal to the free surface, given the body length and the maximum cross-section area. A condition is found that determines whether it is worth replacing a solid of revolution by an equivalent body of three-dimensional shape. The results of the theory are compared for various laws of friction at the contact surface of the body with the medium. Coulomb friction and limit plastic friction.

Among the fundamental problems of penetration theory are the determination of the range of variation of the parameters in which a body of three-dimensional configuration has advantages compared with the traditional axially symmetric shapes in regard to the penetration of dense media, in particular with respect to penetration depth and maximum overloads and also the determination of the optimum shape of such bodies under various isoperimetric conditions.

Irrespective of the remarkable progress achieved in developing models of media and numerical methods (see, e.g. the surveys [1, 2]), theoretical investigation of the motion of non-deformable and deformable bodies in dense media is a problem of extraordinary complexity. Effective solution of various variational problems involving the intrusion of non-deformable bodies into dense media is possible only if the forces acting in the region where the body surface comes into contact with the medium can be expressed explicitly in terms of the body shape, the medium characteristics, and so on. However, no such relationships are available for three-dimensional bodies; exact solutions have been constructed only for the linear problems of slender cyclically symmetric bodies with flat faces entering compressible fluids and elastic media [3–7].

Nevertheless, the lack of rigorous formulae for the forces acting on a three-dimensional body, for specific models of media and modes of motion, does not exclude the possibility of approximately describing such forces using relationships supplied by either approximate or empirical theories. One should note that the substantial spread in the values of the parameters describing the properties of dense media (type of soil and grade of metal), sometimes reaching as much as 20% or more (for example, the cohesion of soil and the plasticity limit), generally precludes any possibility of rigorously specifying the body shape that will maximize or minimize the value of some objective function under given conditions. In applied problems, therefore, it is important to be able to indicate, in principle, whether a penetrating body of axially symmetric or essentially three-dimensional shape should be used in order to achieve, say, the maximum penetration depth for given initial data and for parameters of the medium varying in a certain range.

The optimum three-dimensional shape of bodies of some classes with maximum penetration depth has been investigated numerically [8], using a local interaction model [9] to describe the stresses acting at the contact surface, on the assumption that the shear stresses obey Coulomb's friction law. Examples [8] have shown that replacing a solid of revolution by an equivalent optimum body of three-dimensional shape may substantially increase the penetration depth. However, as the variational problem has been solved by numerical means (method of local variations [10]), no suitable criterion has been developed for such replacement. It would also be very interesting to see how the solution of the problem differs for different models of the shear stress at the body surface.

Using a model of mixed type to describe the shear stress at the surface of a solid of revolution, according to which the stress is determined on the basis of the value of the normal stress, either given by Coulomb's law or equated to the limit plastic friction, Grigoryan [11] demonstrated the extraordinary difficulty of the variational problem of a body of minimum drag [12]. If that model is

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used to maximize the penetration depth of a three-dimensional body, the problem becomes practically unsolvable. In what follows, therefore, this problem will be considered either within the limits of the Coulomb friction model or in terms of the limit plastic friction, when it is assumed in the slender-body approximation that the optimization problem splits into two: for longitudinal and transverse contours.

1. MODELLING OF THE DRAG FORCE ACTING ON A BODY PENETRATING A DENSE MEDIUM ALONG THE NORMAL TO THE FREE SURFACE

Among the experimental publications devoted to the load acting on an intruded body, the paper by Vitman and Stepanov [13] may still be singled out even today, both for its methodology and for its results. Two of its main results deserve special mention: it establishes a two-term model for the load, containing dynamical and strength components; the coefficient of the velocity head in the model is identical with the Newtonian coefficient of pressure on a cone. According to the last result, this coefficient, for which a simple Newtonian computation [14], depending on the shape of the leading part of the body, is available, can be used to compute the penetration depth.

Within the limits of the elastic–plastic model for the material of the obstacle, and on the assumption that when a slender axially symmetric rigid projectile penetrates the material, the latter moves only in layers perpendicular to the projectile axis, independently in each layer (the normal sections hypothesis), a formula has been obtained [15] for the normal stress σ_n at the contact surface, including the velocity and deceleration of the body, the thickness, the inclination and curvature of the longitudinal contour, as well as the characteristics of the medium. Analysis of this solution shows that the apparent additional mass m (the mass of material of the obstacle included in the volume of the leading part, over which the body establishes contact with the medium in a cavitation scheme of flow, or in the volume of the body) occurs in the formula for the penetration depth through a term added to one, whose order of magnitude is determined by the expression $(m/M)t^2$ (where M is the mass of the body and $2t$ is its relative thickness). In the slender-body approximation ($t^2 \ll 1$) this term may be neglected, so that the retardation will not exert a marked influence on the resistance of the body.

If the resistance of the body is computed with allowance for the Coulomb friction force as well, the contribution of the term containing the retardation of the body in the formula for σ_n is represented by two terms added to one: $1 + O(t^2) + O(\mu t)$ (where μ is the coefficient of dry friction). This gives grounds for the assumption that, for realistic μ values, the calculated results remain satisfactorily accurate even if one drops the term depending on the retardation in the formula for the normal stress at the surface of a slender body. This conclusion remains true in the case of constant friction at the contact surface (the model of limit plastic friction). The validity of these estimates is also confirmed by experimental data [13].

Thus, on the basis of the previously obtained solution [15] and the results of its analysis, the formula for the normal stress at the contact surface of the body, neglecting the curvature of the longitudinal contour, may be written as follows:

$$\sigma_n = \frac{1}{2}\rho u^2\left[\ln(1+b) - \frac{b}{1+b}\right](\mathbf{n},\mathbf{x})^2 + \tau[1+\ln(1+b)] > 0 \tag{1.1}$$

where ρ is the density of the material of the obstacle, u is the body velocity, $\mathbf{n}$ is the unit vector normal to the body surface, $\mathbf{x}$ is the unit vector of the x axis, which has its origin at the body tip, coincides with the body axis and is directed against the motion ($(\mathbf{n}, \mathbf{x}) < 0$) and $b = E/[2\tau(1 + \nu)]$, where E, ν and τ are, respectively, Young's modulus, Poisson's ratio and the yield point of the material of the obstacle.

Results have been obtained for rigid axially symmetric bodies penetrating into soils modelled by a plastically compressible medium [16, 17]. Omitting the terms involving the retardation of the body and the curvature of the longitudinal contour—as in the case of collision with a metallic obstacle [15]—the formula for σ_n can be written as

$$\sigma_n = \frac{\rho u^2}{2(1-\gamma)b_1}\left[\varepsilon^{1-\gamma} - 1 + 2(1-\gamma)b_1\varepsilon^{-\gamma} + \frac{(1-\gamma)}{\gamma}(\varepsilon^{-\gamma}-1)\right](\mathbf{n},x)^2 + \\ + \frac{\tau_0}{2\mu_0}(\varepsilon^{-\gamma}-1) + p_a\varepsilon^{-\gamma} > 0 \tag{1.2}$$

$$b_1 = \frac{\rho}{\rho_1},\quad \gamma = \frac{\mu_0}{1+\mu_0},\quad \varepsilon = 1 - b_1,\quad \tau_0 = 2k\cos\vartheta,\quad \mu_0 = \sin\vartheta$$

where ρ_1 is the density of the soil behind the shock wave, k and ϑ are the coefficient of cohesion and angle of internal friction of the soil and p_a is the pressure ahead of the shock wave.

Note that formulae (1.1) and (1.2) for σ_n and the empirical formula described above for the load on the intruded body [13] are identical in nature.

We will now consider the reason that the terms involving the curvature of the longitudinal contour of the body were omitted in (1.1) and (1.2). Allowance for these terms would have made it necessary to introduce the natural restriction $\sigma_n \leqslant 0$, which is akin to the condition, required when the Newton–Busemann formula is used, that the pressure be non-negative [18]. This restriction leads to the appearance of arcs $y(x)$ of the outer extremum ($\sigma_n = 0$), which, according to previous results [15, 17], are solutions of the differential equation

$$A_0 y'^2 + Byy'' + C = 0 \tag{1.3}$$

where the coefficients A_0 and C are determined by (1.1) and (1.2), and the coefficient B has one of the following forms

$$B = \frac{1}{2}\rho u^2 \ln(1+b), \quad B = \frac{\rho u^2}{2b_1\gamma}(\varepsilon^{-\gamma} - 1) \tag{1.4}$$

depending on whether the medium is elastic-plastic or plastically compressible.

The general solution of Eq. (1.3) may be written as

$$x = -2e\sqrt{\frac{A_0}{C}}\int (d_1^2 - z^2)^{e-1/2} dz + d_2, \quad e = \frac{E}{2A_0}, \quad z^2 = d_1^2 - y^{1/e} \tag{1.5}$$

(d_1 and d_2 are arbitrary constants). An outer extremum arc (1.5) may be drawn through the point (0, 0)—the body tip—and the end point $(L, y(L))$ of the contour. In that case, putting $e = 1/2$ to simplify matters (which amounts, e.g., in elastic-plastic media (1.1), (1.4) to assuming that the logarithmic term in square brackets in the expression for A_0 makes the major contribution), we can write solution (1.5) as

$$x = \sqrt{\frac{A_0}{C}}\left(d - \sqrt{d^2 - y^2}\right), \quad d = \frac{1}{2}\sqrt{\frac{C}{A_0}}\left(1 + \frac{A_0}{C}t^2\right)L, \quad t = \frac{y(L)}{L} \tag{1.6}$$

where L is the body length. It is obvious that for solution (1.6) to be meaningful it will suffice that $A_0 t^2/C > 1$. Obviously, a body having the shape of (1.5) or (1.6) (the front part of an ellipsoid of revolution), which is based on the equation $\sigma_n = 0$, has zero drag.

The solution just demonstrated—the outer extremum arc—corresponds to the formation of a cavity. It cannot be considered suitable, not so much because the reason for the formation of a cavity is "indeterminate", but because under these conditions the assumptions adopted previously [15, 17] when the problems were formulated are no longer valid.

Thus, expressions (1.1) and (1.2) for σ_n must be considered as approximations to the solutions of [15, 17], valid when the curvature of the longitudinal contour of the body is small, and also as a means of deriving theoretical estimates of the coefficients in the two-term local interaction model. A similar possibility is offered by the results of [19, 20].

For further treatment, we write the two-term models (1.1) and (1.2) for the normal pressure at the body surface, which contain dynamical and strength terms, in a generalized form

$$\sigma_n = Au^2(\mathbf{n}, \mathbf{x})^2 + C \tag{1.7}$$

We will treat the coefficients A and C as constant parameters of the model, to be determined either theoretically or experimentally.

Since the body velocity in the half-space occupied by the dense medium depends on the initial stage of intrusion, up to the time when contact with the medium takes place over the entire body surface with the normal directed towards the flow, we must evaluate the influence of this stage on the decrease in the initial velocity u_0.

In the case of the Coulomb model of friction, which enables one to obtain a maximum estimate of the required quantity, the force $\mathbf{f}_c$ per unit area of the body surface and the force D resisting its penetration may be written as follows:

$$\mathbf{f}_e = -\sigma_n(\mathbf{n} - \mu\boldsymbol{\tau}), \ (\boldsymbol{\tau}, \mathbf{x}) = |[\mathbf{n}, \mathbf{x}]| \tag{1.8}$$

$$D = \iint_{S_k} (\mathbf{f}_e, \mathbf{x}) dS = -\iint_{S_k} \sigma_n[(\mathbf{n}, \mathbf{x}) - \mu(\boldsymbol{\tau}, \mathbf{x})] dS$$

where S_k is contact surface between the body and the medium.

Let us estimate the velocity drop Δu of an axially symmetric body with conical leading part during the initial stage of penetration into a half-plane occupied by a dense medium along the normal to the free surface. If relations (1.7) and (1.8) are used and it is assumed that $(\mathbf{n}, \mathbf{x}) = -\tau$, $(\tau, x) = 1$ (for slender bodies), the equation of motion of the body takes the form

$$Mh'' = -h^2(At^2h'^2 + C)\left(1 + \frac{\mu}{t}\right)\frac{S_m}{L^2}, \ h' = u \tag{1.9}$$

where h is the actual value of the penetration depth, S_m is the maximum cross-sectional area (the bottom section) of the body. Integrating Eq. (1.9) with initial conditions $h(0) = 0$, $h'(0) = u_0$, we find that $\Delta u/u_0 \equiv [u_0 - h'(L)]/u_0 \ll 1$ if

$$\frac{mAt^2}{M\rho}\left(1 + \frac{\mu}{t}\right)\left(1 + \frac{C}{Au_0^2t^2}\right) \ll 1 \tag{1.10}$$

To satisfy inequality (1.10), it is sufficient that

$$C/(Au_0^2t^2) \leqslant O(1) \tag{1.11}$$

Since A and C occur in (1.11) in the form of a quotient, it follows from (1.1), (1.2) that to estimate its order of magnitude we need only note that $A \sim p$ and $C \sim \tau$ or τ_0. Since $\rho \sim 10^4$ kg/m^3 and $\tau \sim (10^2\text{–}10^3)$ MPa for metallic media, and $\rho \sim 10^3$ kg/m^3 and $\tau_0 \sim 10^{-1}$ MPa for soils, it follows from (1.11) that the velocity drop at the initial stage of incomplete imbedding of the body (or its leading part) in the medium may be neglected without significant loss of accuracy, provided the initial velocity of entry satisfies the inequality $u_0 > (10^2\text{–}10^{5/2})/t$ m/s (for metals) or $u_0 > 10/t$ m/s (for soils).

This result is important for the subsequent definition of the depth-of-penetration functional, since it enables one, under the conditions specified, to ignore the process of incomplete embedding of the body (or its leading part), assuming that while the body is decelerating from a velocity u_0, contact with the medium takes place over its entire surface with the normal pointing in the direction of motion. Thus, if condition (1.11) holds, the expression for the drag force (1.8) may be written as

$$D = -\iint_S \sigma_n[(\mathbf{n}, \mathbf{x}) - \mu(\boldsymbol{\tau}, \mathbf{x})] dS \tag{1.12}$$

where S is the total lateral area of the body, with the normal pointing in the direction of the motion.

Although formula (1.12) was derived for solids of revolution and there are no theoretical grounds for carrying it over to the motion of three-dimensional bodies in a dense medium, we will assume that the local interaction model remains valid in that case too. The admissibility of this assumption may be verified by comparing the penetration depth H in sandy soil for bodies of star-shaped cross-section (Fig. 1) relative to that of solids of revolution equivalent in mass and cross-sectional areas at various distances from the tip [21], determined experimentally, with the results of computations using formulae (1.2), (1.7) and (1.12). Omitting the details, we write down the expression for the ratio H of the depths of penetration of a flat-faced star-shaped body and an equivalent axially symmetric body

$$H = w_1^2 \frac{(1 + w)\ln(1 + w_0^2 / w_1^2)}{(w + w_1)\ln(1 + w_0^2)} \tag{1.13}$$

$$w_1^2 = (1 + a_1^2)a_2, \ a_1 = \operatorname{ctg}\frac{\pi}{n} - \frac{n}{\pi a_2}, \ a_2 = \left(\frac{g_k}{g_0}\right)^2$$

$$g_k^2 = \frac{S_m}{\pi L^2}, \ w = \frac{g_k}{\mu}, \ w_0^2 = \frac{Au_0^2}{C} g_k^2$$

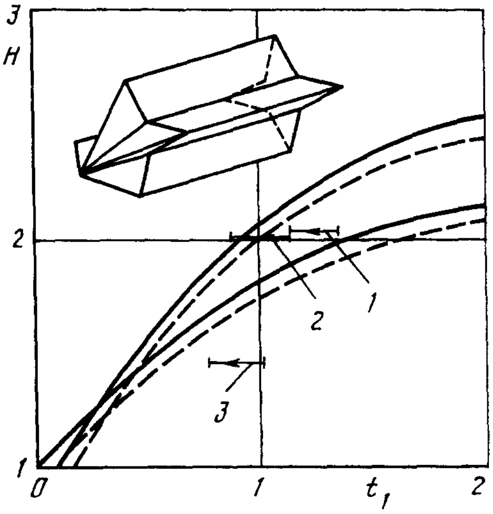

Fig. 1.

where g_k is the radius of the maximum cross-section of an equivalent solid of revolution relative to the length of the leading part and g_0 is the minimum dimensionless radius of the maximum cross-section of a star-shaped body (Fig. 1) consisting of n symmetric cycles. As can be seen, the number of cycles n and the parameter of the cross-sectional shape a_2 affect H through the parameter w_1. If g_k, u_0, μ and the parameters of the medium are given, then, by varying w_1, one can determine the influence of the shape of the transverse contour of a star-shaped body on the relative penetration depth H.

Figure 1 plots H against $t_1 = \lg w_0$ for $w_1 = 1.5$ and 2 (the dashed and solid curves, respectively) and for $w_1 = 1/75$ and 2 (the lower and upper pair of curves, respectively). It should be noted that H is a non-monotonic function of w and w_1 for $t_1 < 0.5$, and also that the relative penetration depth increases as t increases. We also note that when there is an increase in the initial velocity u_0 for which t_1 increases, the value of H ceases to depend on the quotient of logarithmic terms in (1.13), being determined solely by the body geometry and the dry friction coefficient μ in Coulomb's law.

It was assumed in the computations that the sandy soil is sand with a disturbed structure, having the following characteristics: $\rho = 1600\ \text{kg/m}^3$, $\varepsilon = 0.37$, $\mu_0 = 0.5$ and $\tau_0 = 5$ kPa. The backpressure p_a varied in the range $(1–3) \times 10^{-1}$ MPa. The dry friction coefficient was $\mu = 0.2$. The experimental values of H are represented by segments of straight lines in the t_1-intervals corresponding to the indicated variation of p_a. The arrow points in the direction of increasing p_a. The segments 1, 2 and 3 correspond to values of $u_0 = 520$, 310 and 200 m/s. Inequality (1.11) is satisfied in all cases.

Comparison of the experimental and computed H values indicates quite satisfactory qualitative and quantitative agreement, particularly at high initial entry velocities, irrespective of the large range within which p_a varies. The poor agreement between the theoretical and experimental data at $u_0 = 200$ m/s may be explained by the fact that at velocities u_0 less than the propagation velocity of perturbations in the medium, the theory of [17] may no longer be regarded as correct. At such entry velocities one has to use other—e.g. empirical—coefficients A and C in formula (17) for the pressure at the contact surface, defined over the necessary range of velocities u_0.

Summarizing our analysis of the applicability of the two-term local interaction model (1.7) for computing the drag force (1.12) of three-dimensional bodies, we may conclude that it yields computed values of H in excellent agreement with experimental data; it may be used to optimize the shape of three-dimensional bodies penetrating dense media.

2. OPTIMUM SHAPES OF BODIES WHEN THE COULOMB FRICTION LAW IS USED

Let us assume that all the conditions considered in Section 1 are met. We will consider the variational problem of the shape of a slender body with maximum penetration depth, given its length (the length of the leading part) L and the maximum cross-sectional area S_m, in the class of surfaces with similar cross-sections along the x axis, when an analytical solution can be found

$$f \equiv \rho - \varphi(x)\,R(\theta) = 0 \tag{2.1}$$

where (x, ρ, θ) are cylindrical coordinates, and $\varphi(x)$ and $R(\theta)$ are the longitudinal and transverse contours of the body.

Expression (1.12) for the drag of the body when (2.1) is the equation of its surface is conveniently written as

$$D(u) = D_1 u^2 + D_2 \tag{2.2}$$

where D_1 and D_2 may be expressed in the following two forms

$$D_1 = \int_0^L (\gamma_1\dot{\varphi} + \gamma_2)\dot{\varphi}^2\varphi\,dx, \quad D_2 = \gamma_3\int_0^L \varphi\,dx + CS_m \tag{2.3}$$

$$\gamma_1 = AJ_4, \quad \gamma_2 = \mu AJ_3, \quad \gamma_3 = \mu CJ_1$$

$$S_m = \frac{1}{2}\varphi^2(L)J_2; \quad J_n = \int_0^{2\pi} R^n\Phi^{2-n}d\theta, \quad n = 1,2,3,4; \quad \Phi = \sqrt{1+\frac{\dot{R}^2}{R^2}} \tag{2.4}$$

$$D_1 = \int_0^{2\pi} (k_1 R\Phi^{-1} + k_2)R^3\Phi^{-1}d\theta, \quad D_2 = k_3\int_0^{2\pi} R\Phi^{-1}d\theta + CS_m \tag{2.5}$$

$$k_1 = A\int_0^L \varphi\dot{\varphi}^3dx, \quad k_2 = \mu A\int_0^L \varphi\dot{\varphi}^2dx, \quad k_3 = \mu C\int_0^L \varphi dx$$

(the dot denotes differentiation with respect to the argument). This representation of the drag components, depending on the properties of the medium and the shape of the body, justifies splitting the problem of optimizing the body shape into two: determination of the longitudinal and transverse contours, since the variations $\varphi(x)$ and $R(\theta)$ are independent.

Integrating the equation of motion of the body, we find an expression for the penetration depth

$$h = \frac{M}{2D_1}\ln\frac{D_0}{D_2}, \quad D_0 = D(u_0) \tag{2.6}$$

The depth-of-penetration functional h is a functional of non-additive type. Equating its first variation to zero and writing out this condition and the Legendre condition in generalized form, we have

$$\delta D_0 + f\,\delta D_2 = 0 \tag{2.7}$$

$$\delta^2 D_0 + f\,\delta^2 D_2 \geqslant 0 \tag{2.8}$$

Here

$$f = -\frac{I_0/(hD_2) - 1}{I_0/(hD_0) - 1} \equiv -\frac{D_0/D_2 - 1 - \ln(D_0/D_2)}{1 - D_2/D_0 - \ln(D_0/D_2)}, \quad I_0 = Mu_0^2/2 \tag{2.9}$$

where I_0 denotes the initial kinetic energy of the body. In this formulation of the penetration problem, all the body's kinetic energy is used up in the work done by the drag force (2.2) over the path h, the work of the constant component D_2 over that path is $hD_2 < I_0$, and the work of the entire initial drag force is $hD_0 > I_0$. Using these inequalities, we deduce from (2.9) that $f > 0$.

It follows immediately from expression (2.7) for the first variation of the functional h that the shape of a body of minimum resistance, determined at the initial stage of penetration ($\delta D_0 = 0$), is not optimum in the sense of maximizing the penetration depth.

Let us determine the optimum longitudinal contour of the body. Since the basic functions F_0 and F_2 of the functionals D_0 and D_2 (see (2.2) and (2.3)) do not explicitly contain the independent variable x, Euler's equation, which follows from (2.7), admits of a first integral

$$\frac{1}{\gamma_1}\left[\dot{\varphi}\frac{\partial(F_0 + fF_2)}{\partial\dot{\varphi}} - (F_0 + fF_2)\right] \equiv [2\dot{\varphi}^3 + d\dot{\varphi}^2 - (1+f)d_1]\varphi = c_1 \tag{2.10}$$

$$d = \frac{\gamma_2}{\gamma_1}, \quad d_1 = \frac{\gamma_3}{\gamma_1 u_0^2}$$

where c_1 is an arbitrary constant. Without loss of generality, we may assume that the function $\varphi(x)$ satisfied the following boundary conditions at the end points

$$\varphi(0) = 0, \quad \varphi(L) = \sqrt{S_m / \pi} \tag{2.11}$$

Without dwelling on the details, we note that the constant on the right of Eq. (2.10) is non-negative, provided one imposes the natural restriction $\varphi \geqslant 0$ on the shape of the longitudinal contour and takes boundary conditions (2.11) into account: $c_1 \geqslant 0$.

If $c_1 > 0$, the extremal is described by the relations

$$x = \int_0^{\varphi} z^{1/3}[(1+z_1)^{1/3} + (1-z_1)^{1/3}]d\varphi \tag{2.12}$$

$$z = \frac{\varphi}{c_1 + (1+f)d_1\varphi}, \quad z_1 = \sqrt{1 - \left(\frac{d}{3}\right)^3 z}$$

and at the body tip it has the asymptotic form $(32c_1/27)^{1/4}x^{3/4}$.

If $c_1 = 0$, which also corresponds to the case in which the length of the body (of its leading part) L is not given, Eq. (2.10) admits of arcs of an extremal of two types, satisfying the Weierstrass–Erdmann condition at a corner point—the straight lines $\varphi = 0$ and $\varphi = \tau_1 x$, where τ_1 is half the relative thickness of a cone of length L_1, which is defined by the relations

$$\tau_1 = \varphi_1 d_1^{1/3}, \quad 2\varphi_1^3 + d_2\varphi_1^2 - (1+f) = 0, \quad d_2 = \left(\frac{\gamma_2^3 u_0^2}{\gamma_1^2\gamma_3}\right)^{1/3} \tag{2.13}$$

Thus, if the length L is given and $t \leqslant \tau_1$, the optimum body is of length L_1 with a conical longitudinal contour, since a needle—$\varphi = 0$ over the interval $x \in [0, L - L_1]$—has no physical meaning and is assumed to possess a longitudinal contour only in order to satisfy conditions (2.11).

We note that it is convenient to begin the search for the shape of a body with maximum penetration depth with the assumption that the longitudinal contour is conical, $\varphi = \tau_1 x$. Having computed k_i and defined an optimum transverse contour $R(\theta)$, one computes γ_i and then τ_1 from (2.13). If $L_1 \leqslant L$, when the length L is given, the optimum body is thus found. If $L_1 \leqslant L$, when the length L is given, the optimum body is thus found. If $L_1 > L$, one uses expression (2.12) for the longitudinal contour.

To construct an optimum transverse contour, we introduce a new variable

$$r = k_f R(\theta), \quad k_f = (k_1 u_0^2 / k_3)^{1/3} \tag{2.14}$$

Formulae (2.2) for the components of the drag of the body may be rewritten in the form

$$D_1 u_0^2 = \frac{k_3}{k_f}\int_0^{2\pi} (r\Phi_1^{-1} + a)r^3\Phi_1^{-1}d\theta, \quad a = \frac{k_2 k_f}{k_1}, \quad \Phi_1 = \sqrt{1 + \frac{\dot{r}^2}{r^2}} \tag{2.15}$$

$$D_2 = \frac{k_3}{k_f}\int_0^{2\pi} r\Phi_1 d\theta + CS_m \tag{2.16}$$

In the case of a conical longitudinal contour, we have

$$k_f = t\left(\frac{Au_0^2}{\mu C}\right)^{1/3}, \quad a = \left(\frac{\mu^2 A u_0^2}{C}\right)^{1/3} \tag{2.17}$$

We have to find a function $r(\theta)$ satisfying relations (2.7) and (2.8), isoperimetric condition (2.4), which in view of (2.14) becomes

$$2\pi k_f^2 = \int_0^{2\pi} r^2 d\theta \tag{2.18}$$

and the condition that the transverse contour is closed: $r(0) = r(2\pi)$. Formula (2.7), together with isoperimetric condition (2.18), is equivalent to the relation

$$\delta \int_0^{2\pi} (F_0 + \lambda r^2) d\theta + f \delta \int_0^{2\pi} F_2 d\theta = 0 \tag{2.19}$$

where λ is an indefinite constant—a Lagrange multiplier; F_0 and F_2, as in (2.10), are the basic functions of the functionals D_0 and D_2.

The Euler equation obtained from (2.19) has a first integral, which we write in parametric form

$$r^2 = \frac{c_2 + \alpha[2\alpha^3 + a\alpha^2 - (1+f)]}{3\alpha^2 + 2a\alpha + \lambda}, \quad c_2 = \text{const}, \quad \alpha = \frac{r}{\Phi_1}, \quad \alpha \leqslant r \tag{2.20}$$

$$d\theta = \pm \frac{\alpha dr^2}{2r^2\sqrt{r^2 - \alpha^2}}$$

The condition at the corner points becomes

$$\Delta[c_2]\delta\theta + \Delta\left[\frac{\partial(F_0 + \lambda r^2 + fF_2)}{\partial \dot{r}}\right]\delta r = 0 \tag{2.21}$$

Since the positions of the corner points are not given, it follows from (2.21) that $\Delta[c_2] = 0$, and thus all arcs of the extremal have the same value of the constant c_2. Then, if $r(\theta)$ is a solution of Eqs (2.20), so are $r(-\theta)$ and $r(\theta + \text{const})$, since $\dot{r}$ occurs in (2.20) to an even power only, and so the transverse contour may be made up of an integral number n of symmetric cycles. Under those conditions, the condition that the transverse contour be closed will be satisfied.

By (2.8), (2.14)–(2.16) and (2.19)–(2.20), we can write the Legendre condition in the form

$$r^2 \geqslant \frac{1}{2(3\alpha + a)}[8\alpha^3 + 3a\alpha^2 - (1+f)] \tag{2.22}$$

It follows from (2.22) that the optimum body will either be a solid of revolution, or the transverse contour an arc of a circle (an arc of zero inclination, $\dot{r} = 0$), if $r \leqslant \alpha_1$, where α_1 is the only positive root of the equation

$$2\alpha^3 + a\alpha^2 - (1 + f) = 0 \tag{2.23}$$

The restriction on r thus obtained is very strong, and, as already shown in [9], the problem must be reformulated by introducing a differential inequality $\dot{r} \geqslant 0$ and taking into account our previous conclusion that the transverse contour consists of n symmetric cycles. The isoperimetric condition (2.18) becomes

$$\frac{\pi}{n} k_f^2 = \int_0^{\pi/n} r^2 d\theta \tag{2.24}$$

The transversality condition at the ends of the interval $[0, \pi/2]$ may be written as

$$\left[\frac{\partial(F_0 + \lambda r^2 + \lambda_1(\dot{r} - \beta^2) + fF_2)}{\partial \dot{r}}\delta r\right]_0^{\pi/n} = 0 \tag{2.25}$$

where λ_1 is an indefinite variable Lagrange multiplier and β is a function of the variable θ satisfying the differential equation $\dot{r} - \beta^2 = 0$.

Analysis of the necessary conditions for an extremum in the problem of the transverse contour of a body with maximum penetration depth shows that the only difference between them and the similar conditions in the problem of the transverse contour of a body with minimum drag [9] is the presence of the parameter $f > 0$. Hence the proofs of the propositions formulated below are similar to those of the corresponding propositions in [9].

Lemma 1. The extremal of a half-cycle of the transverse contour cannot consist of two regular arcs (2.20) separated by an arc of zero inclination if the maximum radius r_f of the transverse contour is not given.

Corollaries. 1. If the parameter of isometric condition (2.24) satisfies the inequality $k_f < \alpha_1$ and no additional conditions are imposed at the ends of the extremal, the optimum transverse contour is a circle of radius k_f.

2. If the extremal inside a half-cycle of the transverse contour contains an arc of zero inclination, it cannot contain other arcs of non-zero inclination.

3. If the maximum radius r_f of the transverse contour is given, the extremal may contain an arc of zero inclination inside a half-cycle.

Lemma 2. An extremal of a half-cycle of the transverse contour cannot contain, together with a regular arc (2.20), any arc of zero inclination of radius $r_f \geqslant \alpha_1$ or $r_i \geqslant \alpha_1$, if r_f or r_i (the minimum radius of the transverse contour) are not given.

Theorem. If $k_f \geqslant \alpha_1$, when there are no additional restrictions on the extremal of the transverse contour over the interval $[0, \pi/n]$, it may consist solely of an arc of a circle of radius α_1, or of a regular arc $\alpha = \alpha_1$, or both; the existence of each of these structures of an extremal is determined by isoperimetric condition (2.24).

We will call a transverse contour constructed from such arcs of the extremal absolutely optimum. By (2.20), the regular arc $\alpha = \alpha_1$ is a segment of the straight line whose equation is

$$r(\theta) = \alpha_1/\cos(\theta + \gamma) \tag{2.26}$$

It follows from (2.26) that the regular arc touches the circle of radius α_1 at the point $\theta = -\gamma$. If $\gamma = -\theta_c$, where $0 < \theta_c < \pi/n$, then the arc of zero inclination $r_i = \alpha_1$ connects to the regular arc (2.26) without any discontinuity in the derivative.

Using isoperimetric condition (2.24) and Eq. (2.26), we find that

$$\left(\frac{k_f}{\alpha_1}\right)^2 = \frac{n}{\pi}\left[\theta_c + \operatorname{tg}\left(\frac{\pi}{n} - \theta_c\right)\right] \geqslant 1 \tag{2.27}$$

By (2.27), the extremal will be an arc of the circle $r = \alpha_1$ ($\theta_c = \pi/n$) if $k_f = \alpha_1$ and a combination of an arc of zero inclination and a regular arc if

$$\left(\frac{k_f}{\alpha_1}\right)^2 < \frac{n}{\pi}\operatorname{tg}\frac{\pi}{n} \tag{2.28}$$

But if

$$\left(\frac{k_f}{\alpha_1}\right)^2 \geqslant \frac{n}{\pi}\operatorname{tg}\frac{\pi}{n} \tag{2.29}$$

the extremal will consist solely of a regular arc (2.26), where γ is the angle between the normal to the straight line extending the regular arc and the axis of the polar coordinate system; this angle is determined from the relation

$$\left(\frac{k_f}{\alpha_1}\right)^2 = \frac{n}{\pi}\left[\operatorname{tg}\left(\frac{\pi}{n} + \gamma\right) - \operatorname{tg}\gamma\right] \tag{2.30}$$

Thus, the configuration of an absolutely optimum transverse contour is determined by the value of one parameter k_f/α_1, which contains the characteristics of the medium, the entry velocity and the shape of the longitudinal contour. By the theorem, if this parameter exceeds unity one must consider whether it is worth replacing a solid of revolution with a body of an equivalent three-dimensional shape, whose penetration depth is determined through the following corollaries of the theorem.

Corollaries. 1. The penetration depth of a body having an absolutely optimum transverse contour does not depend on the number of cycles n.

In that case, D_1 and D_2, which determine h (2.6), have the form

$$D_1 u_0^2 = \frac{2\pi}{\alpha_1} k_f k_3 (1 + f - \alpha_1^3), \quad D_2 = \frac{2\pi}{\alpha_1} k_f k_3 + CS_m \tag{2.31}$$

2. The penetration depth of a body having a conical longitudinal contour and an absolutely optimum transverse contour does not depend on the relative thickness of the body.

Carrying out the necessary operations using relations (2.17), (2.23) and (2.31), we obtain

$$D_1 u_0^2 = a\alpha_1(\alpha_1 + a)CS_m, \quad D_0 / D_2 = 1 + a\alpha_1^2 \tag{2.32}$$

$$h = \frac{M}{2S_m}\left(\frac{a}{\mu}\right)^2 \frac{\ln(1 + a\alpha_1^2)}{A\alpha_1(\alpha_1 + a)}$$

Note that, by (2.17), (2.23) and (2.32), α_1 is a function of the parameter a only, and hence it depends only on the characteristics of the medium, the entry velocity and the dry friction coefficient. A plot of $\alpha_1(a)$ is shown in Fig. 2. As an example, the graph shows points 1 and 2 plotted with a and α_1 values corresponding to penetration into soil (u_0 = 600 m/s, ρ = 1530 kg/m^3, b_1 = 0.7, ϑ = 20°, k = 0.05 MPa and p_a = 0.3 MPa) and into metal (u_0 = 2000 m/s, ρ = 7600 kg/m^3, E = 0.05 GPa, ν = 0.3 and τ = 1 GPa), respectively, with dry friction coefficient μ = 0.02.

In the case of a conical longitudinal contour, relations (2.13) are identically true, while by (2.23), (2.27) and (2.30) there is a one-to-one correspondence between t and the shape of an absolutely optimum transverse contour with an arbitrary number of cycles n.

Thus, there is a non-denumerable set of absolutely optimum bodies with $t > \mu\alpha_1/a$ which have the same penetration depth h (2.32), conical longitudinal contour and absolutely optimum traverse contour. But if $t \leqslant \mu\alpha_1/a$, the optimum body is a circular cone with penetration depth

$$h_k = \frac{M \ln(1 + Au_0^2 t^2 / C)}{2S_m At(t + \mu)} \tag{2.33}$$

Figure 3 shows the computed penetration depths h (2.32) of an absolutely optimum body, in units of the penetration depth h_k (2.33) of an equivalent cone; the quotient $H = h/h_k$ is plotted against t (the solid curves). Curves 1 and 2 correspond to penetration into soil and metal at velocities u_0 = 600 m/s and u_0 = 2000 m/s, respectively. The segments AB and AC on the straight line H = 1 correspond to t intervals where the optimum body is a circular cone. Plotted there are computed data of the ratio of the maximum to the minimum radii of an absolutely optimum transverse contour—$R_1 = R(\pi/n)/R(0)$ (the dashed curves). It can be seen that the parameter R_1 is very large in the case of media with low resistance to deformation (soil). Since such a configuration of the transverse contour may be of no practical interest, when $k_f > \alpha_1$ one must look for an optimum transverse contour with restrictions. One such restriction, in particular, is to specify a minimum radius $r_i > \alpha_1$.

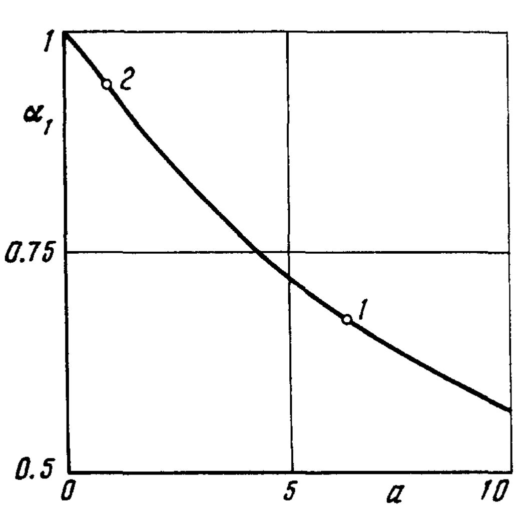

Fig. 2.

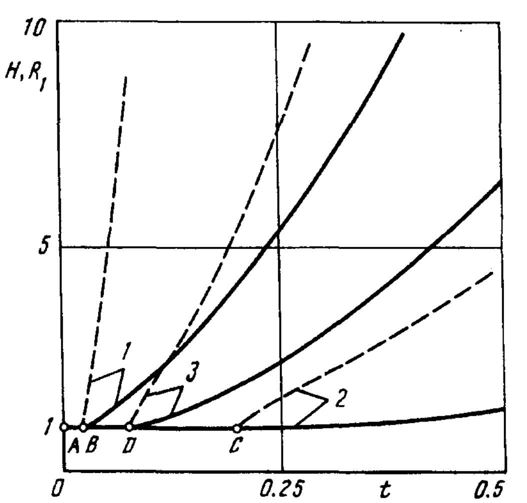

Fig. 3.

The system of relations that in this case defines the parameters of the point of contact of an optimum transverse contour consisting of an arc of zero inclination $r = r_i$ over the interval $\theta \in [0, \theta_c]$ and a concave regular arc (2.20) over the interval $\theta \in [\theta_c, \pi/n]$ has the form

$$c_2 - \lambda r_i^2 = r_i^4 + a r_i^3 + (1+f) r_i \tag{2.34}$$
$$c_2 - \lambda r_i^2 = \alpha_c^2(3r_i^2 - 2\alpha_c^2) + a\alpha_c(2r_i^2 - \alpha_c^2) + (1+f)\alpha_c$$
$$\frac{\pi}{n} = \theta_c + \int_{\alpha_c}^{\alpha_f} \frac{Q}{r^2} d\alpha, \quad Q = \frac{\alpha}{2\sqrt{r^2 - \alpha^2}} \frac{dr^2}{d\alpha}$$
$$K^2 = \frac{\pi}{n}\left(\theta_c + \frac{1}{r_i^2}\int_{\alpha_c}^{\alpha_f} Q d\alpha\right)^{-1}$$

where $K = r_i/k_f = R(0) \leqslant 1$ is a parameter representing the shape of the transverse contour; r_i, n and k_f are given, while c_2, λ, α_c and θ_c are unknown quantities; $\alpha_f = \alpha_1$, as follows from transversality condition (2.25) [9]; according to (2.9), (2.15), (2.16) and (2.20), f depends on both the given and the unknown quantities. A condition for the realization of this structure of the extremal is

$$\alpha_1/k_f \leqslant K \leqslant 1 \tag{2.35}$$

When $\theta_c = 0$, when the composite extremal degenerates into a regular arc, Eqs (2.34) impose a restriction on the defining parameters. Passage from one structure of the extremal to another may be specified by using (2.28) and (2.29). If inequality (2.28) is true and K, while decreasing, becomes equal to α_1/k_f (see (2.35)), one has the case of an absolutely optimum transverse contour consisting of an arc of zero inclination and a regular arc—a straight-line segment. If inequality (2.29) holds and $K = K_n \equiv \alpha_1/(k_f \cos\gamma) < 1$, with K_n defined in accordance with Eq. (2.26), one again has the case of an absolutely optimum extremal—a straight-line segment. But if the defining parameters of the problem are such as to satisfy the inequalities (2.28) and $K < \alpha_1/k_f$, or the inequalities (2.29) and $K < K_n$, then the extremal is convex.

As an example, Fig. 4 shows curves in the plane of the parameters $R_0 = R(0)$ and t, separating regions with qualitatively different shapes of the extremal in a half-cycle of the transverse contour (hatched), in the case of a body with conical longitudinal contour and $n = 4$ cycles penetrating soil at velocity $u_0 = 600$ m/s (curves 1) and metal at $u_0 = 2000$ m/s (curves 2). The segments ME of the straight line $R_0 = 1$ correspond to a circular cone, which is an optimum body at the corresponding t values. If E is the point on curves 2 (metal) and, in particular, curves *1* (soil) at which the transition to bodies of three-dimensional shape occurs, then the ordinate of this point is small; the same has been observed in the problem of bodies of minimum drag [9]. The segments of hyperbolae EG correspond to an absolutely optimum contour consisting of the arc $r = \alpha_1$ and a straight-line segment. Curves 1 and 2, beginning at points G, correspond to an absolutely optimum contour—a straight-line segment (the left curves)—and to transition from an extremal consisting of a concave regular curve to a combination of an arc of zero inclination and a concave regular arc (the right curves).

3. OPTIMUM SHAPES OF BODIES WHEN THE SURFACE FRICTION EQUALS THE MAXIMUM SHEARING STRESS OF THE MATERIAL OF THE MEDIUM

The previous analysis (Section 2) of the problem of the shape of a three-dimensional body with maximum penetration depth, on the assumption that the Coulomb friction law holds at the surface of contact of the body with the medium, carries over without essential changes to the case of a constant shear stress on the surface, equal to the plastic shearing stress of the material of the medium. We will indicate the main differences in the solution of the variational problem that arise when use is made of a model with constant shear stress σ_s on the contact surface.

In this case, the drag of the body may be written in the form

$$D = \iint_S [-\sigma_n(\mathbf{n}, \mathbf{x}) + \sigma_s(\boldsymbol{\tau}, \mathbf{x})] dS$$

Relations (2.3), (2.5), (2.10), (2.13), (2.15)–(2.17), (2.20), (2.22) and (2.23) are therefore considered

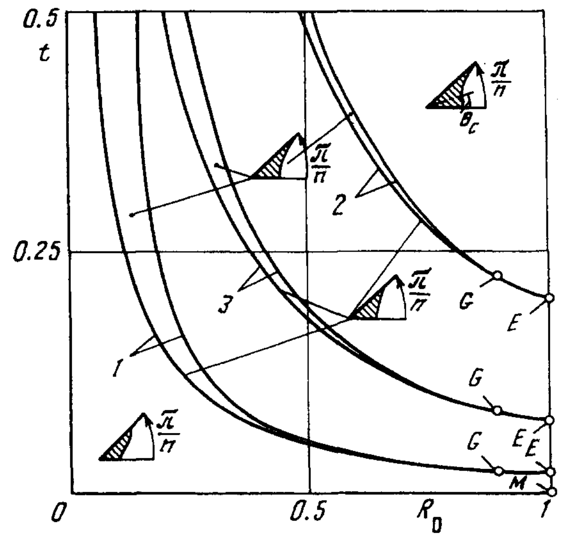

Fig. 4.

with

$$\gamma_2 = k_2 = d = d_2 = a = 0 \tag{3.1}$$

$$\gamma_3 = \sigma_s \int_0^{2\pi} R\Phi d\theta, \quad k_3 = \sigma_s \int_0^{L} \varphi dx$$

The formulae for γ_1 (2.3) and k_1 (2.5) remain unchanged.

Since $d = d_2 = 0$ (3.1), the formula for the optimum longitudinal contour is simplified when $c_1 > 0$ (2.12)

$$x = 2^{1/3} \int_0^{\varphi} \left(\frac{\varphi}{c_1 + d_1\varphi} \right)^{1/3} d\varphi$$

and when $c_1 = 0$ (2.13) the relative thickness of the optimum cone is defined by an explicit expression

$$\tau_1 = [d_1(1+f)/2]^{1/3}$$

Since $a = 0$ (3.1), we deduce directly from (2.23) that

$$\alpha_1 = [(1+f)/2]^{1/3} \tag{3.2}$$

All statements concerning the structure of an extremal of the transverse contour in the model assuming Coulomb friction at the contact surface (Section 2) remain valid in this case too. Relations (2.31) retain the same form (Corollary 2 of the theorem), but relations (2.32) become

$$D_1 u_0^2 = \alpha_1^2 \sigma_s S_m \left(\frac{A u_0^2}{\sigma_s} \right)^{1/3} \tag{3.3}$$

$$\frac{D_0}{D_2} = 1 + \frac{g\alpha_1^3}{g+\alpha_1}, \quad g = \frac{\sigma_s k_f}{Ct}, \quad k_f = t\left(\frac{A u_0^2}{\sigma_s} \right)^{1/3}$$

By the expression for D_0/D_2 (3.3), as well as (2.9) and (3.2), in the case of bodies with conical longitudinal contour and absolutely optimum transverse contour, the parameter α_1 occurring in the relation $k_f = \alpha_1$ determining the transition from solids of revolution to bodies of three-dimensional shape depends solely on the parameter g.

By (3.1), relations (2.34) for the optimum transverse contour, consisting of arcs of zero inclination if a given radius and regular arcs, are also simplified.

The solid and dashed curves 3 in Fig. 3 represent the relative values of the penetration depth $H(t)$ of an absolutely optimum body and particular maximum and minimum dimensions $R_1(t)$ of its transverse contour (for $n = 4$ cycles), penetrating soil at a velocity $u_0 = 600$ m/s on the assumption that $\sigma_s = 0.3$ MPa. As before, the segment AD of the straight line $H = 1$ corresponds to the range of t over which, under the conditions stipulated, the optimum body is a cone. In Fig. 4, curves 3, as in the case of Coulomb friction, show the domains of existence and boundaries of different structures of the transverse contour of the optimum body, consisting of four cycles, with linear longitudinal contour, for penetration of soil. The data presented indicate that the model of constant friction at the contact surface implies no qualitative modifications of the results obtained when the analogous variational problem is solved using the dry friction model.

4. EXAMPLES OF COMPUTATIONS

Figure 5 represents results of a computation of the relative penetration depth H for a conical body with $n = 4$ cycles, for soil ($u_0 = 600$ m/s, curves 1) and metal ($u_0 = 2000$ m/s, curves 2), obtained using the Coulomb friction model. Results are also shown for the constant limit plastic friction model (soil, curves 3). The results are shown for $t = 1/3$, plotted against the dimensionless radius tR_0 of the transverse contour. The solid curves correspond to the optimum transverse contour, the dashed curves correspond to an equivalent flat-faced body and the dash-dot line is the straight line $H = 1$. The maximum possible gain in penetration depth achieved by changing to a body of three-dimensional shape from an equivalent circular cone with $t = 1/3$, for given media and entry velocities is indicated by the points at which the curves touch one another, where the flat-faced body is a body with absolutely optimum transverse contour.

Figure 6 presents data relating to the quantity $R_1^{-1}(tR_0)$ for conical bodies with $t = 1/3$ consisting of four cycles, with optimum transverse contour, as given by the Coulomb friction model (curve 1, $\mu = 0.2$) and the limit plastic friction model (curve 2, $\sigma_s = 0.3$ Mpa) at the surface of contact, for penetration of soil ($u_0 = 600$ m/s). Curve 3 corresponds to equivalent flat-faced bodies. The abscissae of the points of intersection of the latter with curves 1 and 2 coincide with those of the points of contact of the solid and dashed curves 1 and 3 in Fig. 5.

We also show in Fig. 6, as an example, the contours of the cross-sections of equivalent bodies in one cycle ($n = 4$): a circular cone (the arc of a circle, $t = 1/3$), of optimum bodies with linear longitudinal contour penetrating soil (contour 1 for the Coulomb friction model and contour 2 for the constant plastic friction model), and of a flat-faced body with $tR_0 = 0.3$. The relative penetration depth H for bodies with contours 1 and 2 is 1.47 and 1.43, respectively; for a flat-faced body, in the dry friction and plastic friction models, it is 1.14 and 1.15, respectively. The indicated H values were obtained by computing the penetration depth of an equivalent cone using the appropriate friction models.

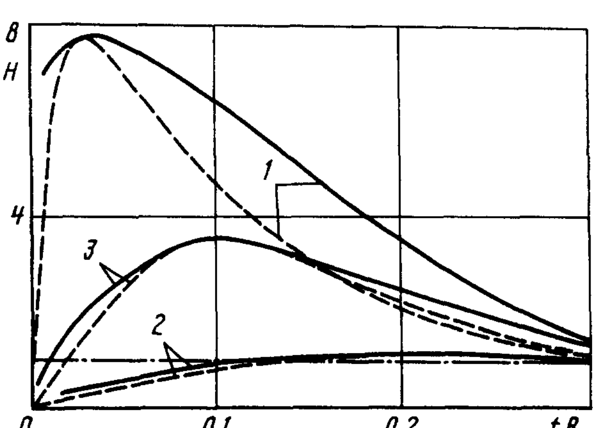

Fig. 5.

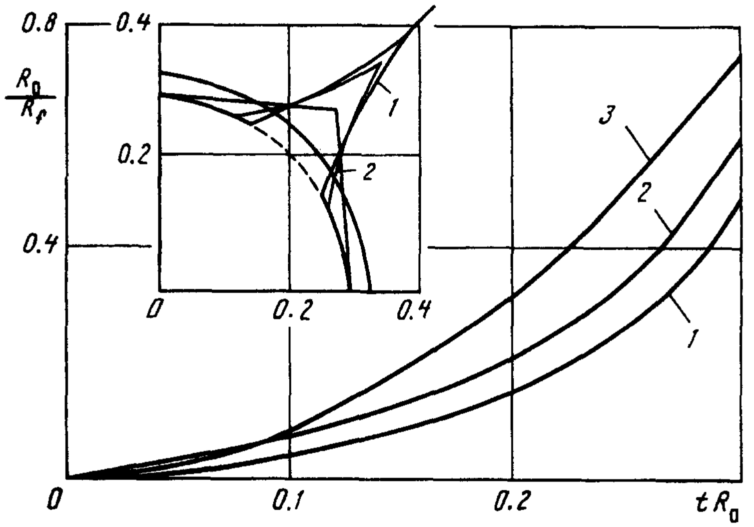

Fig. 6.

In conclusion, it must be noted that the computed results presented in this and the previous sections for the penetration depth and other quantities, attributed to various optimum bodies, should not be seen as corresponding to any actual medium. The values assumed for the defining parameters in (1.1) and (1.2), as well as σ_s, resemble the characteristic values for soils and metals only in order of magnitude. Accordingly, the computed data only demonstrate the advantage of three-dimensional shapes compared with the equivalent solids of revolution, as regards penetration depth in media with essentially distinct characteristics, as well as the possibilities of the theory. The penetration depths of optimum three-dimensional bodies, obtained in the context of the different friction models (Sections 2 and 3), could be compared only as a result of numerical calculations of penetration for bodies with given geometry, using a mixed model of friction at the surface of contact of the body with the medium [11, 12].

computation of H for an optimum body penetrating metal at a velocity u_0 = 2000 m/s (Fig. 5, curves 2) should also be treated with a critical eye. At such velocities, the assumption that the body is not deformed cannot be taken for granted. Also, taking into consideration the very slight increase in penetration depth for an absolutely optimum body compared with an equivalent cone (Fig. 3, solid curve 2), one may therefore conclude that in the case of penetration into metal the transition from a solid of revolution to an equivalent optimum body of three-dimensional shape will most likely not yield any substantial change in the penetration depth. For penetration into soil, however, the transition may yield a significant increase in penetration depth in the approximate domain of parameter values (Fig. 5, curves 1, 3).

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REFERENCES

1. GORSHKOV, A. G. and TARLAKOVSKII, D. V., Dynamical contact problems for a deformable half-space. In *Advances in Science and Technology. Ser. Mechanics of Deformed Solids.* Vsesoyuz. Inst. Nauch. Tekhn. Inform., Moscow, 1990, 76–131.
2. APTUKOV, V. N., Penetration: Mechanical aspects and mathematical modelling. *Problemy Prochnosti*, 1990, 2, 60–68.
3. GONOR, A. L., The motion of bodies with star-shaped cross-section in a compressible liquid with a free surface. In *Problems of Modern Mechanics*, Pt 1. Izd. Mosk. Gos. Univ., Moscow, 1983, 101–112.
4. OSTAPENKO, N. A., Penetration of a slender body with star-shaped bodies into a compressible liquid. *Prikl. Mat. Mekh.*, 1989, **307**, 1, 62–66.
5. GONOR, A. L. and PORUCHIKOV, V. B., Penetration of star-shaped bodies into a compressible liquid. *Prikl. Mat. Mekh.*, 1989, **53**, 3, 405–412.
6. GONOR, A. L., OSTAPENKO, N. A., PORUCHIKOV, V. B. and CHERNYI, G. G., Slender body entry into a compressible fluid. In *Fluid Mechanics. Mechanical Engineering and Applied Mechanics*, 1991, **2**, 175–192.
7. OSTAPENKO, N. A., Penetration of a thin cyclically symmetric three-dimensional body into an elastic half-space. *Prikl. Mat. Mekh.*, 1991, **55**, 5, 808–818.
8. OSTAPENKO, N. A., ROMANCHENKO, V. I. and YAKUNINA, G. Ye., Optimum shapes of three-dimensional bodies with maximum penetration depth in dense media. *Zh. Prikl. Mekh. Tekhn. Fiz.*, 1994, 4, 32–40.

9. OSTAPENKO, N. A. and YAKUNINA, G. Ye., On bodies of least drag moving in media in the case of a locality law. *Izd. Ross. Akad. Nauk. MZhG*, 1992, 1, 95–106.
10. CHERNOUS'KO, F. L. and BANICHUK, N. V., *Variational Problems of Mechanics and Control*. Nauka, Moscow, 1973.
11. GRIGORYAN, S. S., A new friction law and a mechanism of large-scale avalanches and landslides. *Dokl. Akad. Nauk SSSR*, 1979, **244**, 4, 846–849.
12. OSTAPENKO, N. A., The theory of slender solids of revolution of minimum drag moving in dense media with a mixed law of friction at the contact surface. In *Problems of Modern Mechanics* (Edited by S. S. Grigoryan). Izd. Mosk. Gos. Univ., Moscow, 1989, 168–178.
13. VITMAN, F. F. and STEPANOV, V. A., The effect of deformation rate on the resistance to deformation of metals at impact velocities of 10^2–10^3 m/s. In *Some Problems in the Strength of Rigid Bodies*. Izd. Akad. Nauk SSSR, Moscow and Leningrad, 1959, 207–221.
14. VITMAN, F. F. and SLATIN, N. A., The collision of deformable bodies and its simulation. *Zh. Tekh. Fiz.*, 1963, **33**, 8, 982–989.
15. SAGOMONYAN, A. Ya., Piercing of a plate by a thin solid projectile. *Vestnik Mosk. Gos. Univ., Ser. 1: Matematika, Mekhanika*, 1975, 5, 104–111.
16. RAKHMATULIN, Kh. A., SAGOMONYAN, A. Ya. and ALEKSEYEV, N. A., *Problems in Soil Dynamics*. Izd. Mosk. Gos. Univ., Moscow, 1964.
17. SAGOMONYAN, A. Ya., *Penetration*. Izd. Mosk. Gos. Univ., Moscow, 1974.
18. MIELE, A. (Ed.), *Theory of Optimum Aerodynamic Shapes*. London, Academic Press, 1965.
19. FLITMAN, L. M., The boundary layer in some problems of dynamics of a plastic medium. *Izv. Akad. Nauk SSSR. MTT*, 1982, 1, 131–137.
20. FLITMAN, L. M., Subsonic axially symmetric elastic-plastic flow past thin tapered solids of revolution. *Izv. Akad. Nauk SSSR. MTT*, 1991, 4, 155–164.
21. BONDARCHUK, V. S., VEDERNIKOV, Yu. A., DULOV, V. G. and MININ, V. F., The optimization of star-shaped pellets. *Izv. Sibirsk. Otd. Akad. Nauk SSSR. Ser. Tekhn. Nauk*, 1982, **13**, 3, 50–65.

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